ISRAEL JOURNAL OF MATHEMATICS, Vol. 56, No. 2, 1986

ARITHMETICAL CONDITIONS ON THE CONJUGACY VECTOR OF A FINITE GROUP

BY
ANTONIO VERA LÓPEZ
Departamento de Matemáticas, Facultad de Ciencias, Universidad del Pais Vasco, Apartado 644 Bilbao, Spain

ABSTRACT

We get new properties of the numbers $r_G(xN) = |\{\mathrm{Cl}_G(g) \mid \mathrm{Cl}_G(g) \cap xN \neq \emptyset\}$ (where G is a finite group and N is a normal subgroup of G) that are useful in the analysis of the classification of the finite groups according to the number of conjugacy classes.

In the following, G will denote a finite group and we will use the standard notation of the theory of groups. Moreover, if S is a non-empty subset of G, we define

$$r_G(S) = |\{\mathrm{Cl}_G(g) \mid \mathrm{Cl}_G(g) \cap S \neq \emptyset\}|.$$

Naturally, $r_G(S)$ is the number of conjugacy classes that make up the normal set $\bigcup_{g \in G} S^g$. In particular, $r(G)$ denotes the number $r_G(G)$ of conjugacy classes of elements of G. Also, if S is a normal set in G, and

$$S = \mathrm{Cl}_G(z_1) \dot{\cup} \cdots \dot{\cup} \mathrm{Cl}_G(z_t) \qquad \text{with } |C_G(z_1)| \geqq \cdots \geqq |C_G(z_t)|,$$

we define $\Delta_S^G = (|C_G(z_1)|, \ldots, |C_G(z_t)|)$ and the r-tuple $\Delta_G^G = \Delta_G$ will be called the conjugacy vector of G.

In |4|, we saw that if $N \trianglelefteq G$ and $\{\bar{g}_1 = \bar{1}, \bar{g}_2, \ldots, \bar{g}_t\}$ is a set of representatives from the conjugacy classes of $\bar{G} = G/N$, then $r(G) = r_G(N) + \Sigma_{i=2}^t r_G(g_iN)$, and we analyze the number $r(G)$ through the analysis of the numbers $r_G(g_iN)$. In fact, we obtain information about $r(G)$ and Δ_G, once they have been fixed, either arithmetical conditions about $|C_{\bar{G}}(\bar{g}_i)|$, or the structure of $C_{\bar{G}}(\bar{g}_i)$. These results will be used later on, as auxiliary lemmas to classify finite groups according to the number of conjugacy classes. The aim of this note is to get new properties of the numbers $r_G(g_iN)$, that are useful in the analysis of the classification of the finite groups according to the number of conjugacy classes.

Received December 1, 1985

1. The number $r_G(gN)$

In the following, whenever N will denote a normal subgroup of G, $\bar{G} = G/N$ and $T_g = \bigcup_{x\in G} g^x N$. Evidently, we have

$\mathrm{Cl}_{\bar{G}}(\bar{y}_1) = \mathrm{Cl}_{\bar{G}}(\bar{y}_2)$ if and only if $T_{y_1} = T_{y_2}$ and $|\{T_g \mid g \in G\}| = r(G/N) = t$.

Let $\{\bar{g}_1 = \bar{1}, \bar{g}_2, \ldots, \bar{g}_t\}$ be a set of representatives from the conjugacy classes of G/N, then we have

$$(1) \qquad r(G) = r_G(N) + \sum_{i=2}^{t} r_G(g_i N).$$

Next, we will analyze the values $r_G(gN)$ for $g \in G - N$.

Lemma 1. *Let $N \trianglelefteq G$ and z, y be two elements of G. Then $|C_G(y) \cap zN| \neq 0$ iff $zN \in C_G(y)N/N = \overline{C_G(y)}$. Further, in this case, $|C_G(y) \cap zN| = |C_N(y)|$.*

Proof. If $|C_G(y) \cap zN| \neq 0$, then there exists $x \in C_G(y) \cap zN$, so $[x, y] = 1$ and $zN = xN \in \overline{C_G(x)}$. Therefore

$$|C_G(y) \cap zN| = |C_G(y) \cap xN| = |C_G(y) \cap N| = |C_N(y)|,$$

because x commutes with y. Reciprocally, if $zN \in \overline{C_G(y)}$, then $z = xn$, with $x \in C_G(y)$, $n \in N$. Hence $zN = xN$ and $|C_G(y) \cap zN| = |C_G(y) \cap xN| = |C_N(y)|$.

Theorem 1. *Let $N \trianglelefteq G$. Then for each element g of G, we have*

$$r_G(gN) = (1/|G|) \cdot \sum_{x\in G} |\mathrm{Cl}_{\bar{G}}(\bar{g}) \cap \overline{C_G(x)}| \cdot |C_N(x)|$$

where $\bar{G} = G/N$ and $\overline{C_G(x)} = C_G(x)N/N$ for each $x \in G$.

Proof. The number $r_G(gN) = r_G(T_g)$ is the number of orbits from the action by conjugation of G over T_g, $f\colon G \to \Sigma_{T_g}$, given by

$$f(x)(w) = w^x = x^{-1}wx \qquad \text{for every } w \in T_g.$$

Set

$$A = \{(w, x) \in T_g x G \mid w^x = w\}.$$

Then, we have $|G| \cdot r_G(gN) = |A| = \Sigma_{x\in G}\, \theta(x)$, with

$$\theta(x) = |\{w \in T_g \mid w^x = w\}|.$$

Let $\{\bar{y}_1, \ldots, \bar{y}_e\}$ be a system of representatives for the right cosets of $C_{\bar{G}}(\bar{g})$

in $\bar{G}$. Then $T_g = \dot{\bigcup}_{j=1}^{e} g^{y_j}N$ and

$$\theta(x) = \left| C_G(x) \cap \left(\dot{\bigcup_{j=1}^{e}} g^{y_j}N \right) \right| = \sum_{j=1}^{e} | C_G(x) \cap g^{y_j}N |.$$

But

$$| C_G(x) \cap g^{y_j}N | = 0 \qquad \text{if } \bar{g}^{\bar{y}_j} \notin \overline{C_G(x)}$$

(by Lemma 1), therefore it follows that

$$r_G(gN) = (1/|G|) \cdot \sum_{x \in G} \theta(x)$$

$$= (1/|G|) \cdot \sum_{x \in G} \left(\sum_{\bar{g}^{\bar{y}_j} \in \overline{C_G(x)}} | C_G(x) \cap g^{y_j}N | \right).$$

Moreover $\mathrm{Cl}_{\bar{G}}(\bar{g}) = \{\bar{g}^{\bar{y}_1}, \ldots, \bar{g}^{\bar{y}_e}\}$, so that Lemma 1 gives

$$r_G(gN) = (1/|G|) \cdot \sum_{x \in G} |\mathrm{Cl}_{\bar{G}}(\bar{g}) \cap \overline{C_G(x)}| \cdot |C_N(x)|.$$

Corollary 1. *Let j be an integral number such that* g.c.d.$(j, |G|) = 1$. *Then* $r_G(gN) = r_G(g^jN)$. *In particular, we have* $r_G(gN) = r_G(g^{-1}N)$.

Proof. Since j is a number coprime to $|G|$, we have $C_G(x) = C_G(x^j)$ and $C_N(x) = C_N(x^j)$ for each $x \in G$, therefore it is an immediate consequence of Theorem 1.

Definition. Given two elements $y, z \in G$, we say that y is N-conjugate to z, if there exists $n \in N$ such that $z = y^n = n^{-1}yn$. The equivalence N-classes are denoted by $x^N = \{x^n \mid n \in N\}$.

To analyze the cardinal $|\mathrm{Cl}_{\bar{G}}(\bar{g}) \cap \overline{C_G(x)}|$, we observe that $\bar{y} \in \overline{C_G(x)}$ if and only if y fix the N-class x^N, by conjugation, that is $(x^N)^y = x^N$. If u is the number of conjugacy N-classes in G, we have $u \cdot |N| = \sum_{n \in N} |C_G(n)|$, so

$$u = (1/|N|) \cdot \sum_{n \in N} |C_G(n)| = |G/N| \cdot (1/|G|) \cdot \sum_{n \in N} |C_G(n)| = |G/N| \cdot r_G(N).$$

Corollary 2. *Let $N \trianglelefteq G$ and $g \in G - N$ be such that $\bar{g} \in Z(\bar{G})$. Then $r_G(gN) = r/|G/N|$, where r is the number of distinct N-classes in G that are fixed by the automorphism $\alpha_g\colon G \to G$ defined by $\alpha_g(x) = x^g$ $\forall x \in G$. In consequence $r_G(gN) \leqq r_G(N)$ and we have the equality if and only if $\bar{g} \in \overline{C_G(x)}$ for each $x \in G$.*

PROOF. We have $\mathrm{Cl}_{\bar{G}}(\bar{g}) = \{\bar{g}\}$, therefore

$$r_G(gN) = (1/|G|)\cdot \Sigma\{|C_N(x)| \mid x \in G \text{ and } \bar{g} \in \overline{C_G(x)}\}.$$

We know that $\bar{g} \in \overline{C_G(x)}$ iff $(x^N)^g = x^N$. Set $z_1^N, \ldots, z_r^N$ the different conjugacy N-classes that are fixed by α_g. If $x \in G - (z_1^N \dot{\cup} \cdots \dot{\cup} z_r^N)$, then $\bar{g} \notin \overline{C_G(x)}$, so

$$r_G(gN) = (1/|G|) \sum_{i=1}^{r} \sum_{x \in z_i^N} |C_N(x)|;$$

but $|C_N(z_i^n)| = |(C_N(z_i))^n| = |C_N(z_i)|$ and $|z_i^N| = |N : C_N(z_i)|$, therefore

$$r_G(gN) = (1//|G|)\cdot \sum_{i=1}^{r} |N: C_N(z_i)|\cdot|C_N(z_i)| = (|N|/|G|)\cdot r.$$

Finally, as the number of N-classes is $r_G(N)\cdot|G/N|$, we obtain $r_G(gN) \leqq r_G(N)$.

COROLLARY 3. *Let $N \trianglelefteq G$ and $g \in G - N$ be such that $\bar{g} \in Z(\bar{G})$. If* g.c.d.$(j, o(\bar{g})) = 1$, *then* $r_G(g^jN) = r_G(gN)$.

PROOF. We have $r_G(gN) = (1/|G|)\cdot \Sigma\{|C_N(x)| \mid x \in G$ and $\bar{g} \in \overline{C_G(x)}\}$. As g.c.d.$(j, o(\bar{g})) = 1$, we have $\bar{g} \in \overline{C_G(x)}$ iff $\bar{g}^j \in \overline{C_G(x)}$, hence $r_G(gN) = r_G(g^jN)$.

EXAMPLES. In particular, Corollaries 2 and 3 are verified for each $g \in G - N$, if G/N is an abelian group.

COROLLARY 4 (Burnside). *Let $N \trianglelefteq G$. Suppose that $G/N = \langle \bar{g} \rangle \simeq C_p$, with p a prime number. Then:*

(1) $r_G(g^jN) = s$ *for each $j = 1, 2, \ldots, p-1$, where s is the number of conjugacy classes of N fixed by the automorphism $\psi_g : N \to N$ defined by $\psi_g(n) = n^g$ for each $n \in N$;*

(2) $r(G) = ps + (r(N) - s)/p$.

PROOF. It is obvious that $r_G(g^jN) = r_G(gN)$ for each $j = 1, 2, \ldots, p-1$, and consequently

$$r(G) = r_G(N) + (p-1)r_G(gN).$$

Moreover $r_G(N) = s + (r(N) - s)/p$, because, if $\mathrm{Cl}_N(n)^g = \mathrm{Cl}_N(n)$, then $\mathrm{Cl}_G(n) = \mathrm{Cl}_N(n)$, and if $\mathrm{Cl}_N(m)^g \neq \mathrm{Cl}_N(m)$, we have

$$\mathrm{Cl}_G(m) = \mathrm{Cl}_N(m) \dot{\cup} \mathrm{Cl}_N(m^g) \dot{\cup} \cdots \dot{\cup} \mathrm{Cl}_N(m^{g^{p-1}}).$$

On the other hand, $r_G(gN) = (1/|G|)\cdot \Sigma\{|C_N(x)| \mid x \in G$ and $\bar{g} \in \overline{C_G(x)}\}$ and if

$x \in G - N$, we have $\bar{1} \neq \bar{x} \in \overline{C_G(x)}$, so $\overline{C_G(x)} = \bar{G}$ and $g \in \bar{G}$. Thus

$$r_G(gN) = (1/|G|) \cdot \Bigg(\sum_{\substack{n \in N \\ \bar{g} \in \overline{C_G(n)}}} |C_N(n)| + \sum_{x \in G-N} |C_N(x)| \Bigg).$$

Moreover, if $\mathrm{Cl}_N(n_1), \ldots, \mathrm{Cl}_N(n_s)$ are the conjugacy classes of N fixed by ψ_g, as $\bar{g} \in \overline{C_G(n)}$ iff $\mathrm{Cl}_N(n)^g = \mathrm{Cl}_N(n)$, it follows that

$$\sum_{\substack{n \in N \\ \bar{g} \in \overline{C_G(n)}}} |C_N(n)| = \sum_{i=1}^{s} \sum_{n \in \mathrm{Cl}_N(n_i)} |C_N(n)| = \sum_{i=1}^{s} \mathrm{Cl}_N(n_i)| \cdot |C_N(n_i)| = |N| \cdot s,$$

therefore

$$\begin{aligned} r_G(gN) &= (1/|G|) \cdot \Bigg(|N| s + \sum_{x \in G-N} |C_N(x)| \Bigg) \\ &= (1/|G|) \cdot \Bigg(|N| s + \sum_{x \in G} |C_N(x)| - \sum_{n \in N} |C_N(n)| \Bigg) \\ &= (1/|G|) \cdot (|N| s + r_G(N) \cdot |G| - r(N) \cdot |N|) \\ &= s/p + (r_G(N) - r(N)/p) \\ &= s/p + (s - s/p) \\ &= s. \end{aligned}$$

COROLLARY 5. *Let* $N \trianglelefteq G$ *and* $g \in G - N$. *Suppose that* $\mathrm{Cl}_N(m_1), \ldots, \mathrm{Cl}_N(m_\lambda)$ *are the conjugate classes of elements of* N *that are fixed by some conjugate of* g *in* G. *Then*

$r_G(gN) =$

$$(1/|G|) \cdot \Bigg(\sum_{i=1}^{\lambda} |\mathrm{Cl}_{\bar{G}}(\bar{g}) \cap \overline{C_G(m_i)}| \, |N| + \sum_{x \in G-N} |\mathrm{Cl}_{\bar{G}}(\bar{g}) \cap \overline{C_G(x)}| \, |C_N(x)| \Bigg).$$

PROOF. Let's observe

$$|\mathrm{Cl}_{\bar{G}}(\bar{g}) \cap \overline{C_G(m)}| = 0 \qquad \text{for each } m \in N - \Bigg(\dot{\bigcup_{i=1}^{\lambda}} \mathrm{Cl}_N(m_i) \Bigg),$$

since $\mathrm{Cl}_N(m) g^z \neq \mathrm{Cl}_N(m)$ for each $g^z \in \mathrm{Cl}_G(g)$ is equivalent to $\bar{g}^{\bar{z}} \notin \overline{C_G(m)}$ for each $\bar{g}^{\bar{z}} \in \mathrm{Cl}_{\bar{G}}(\bar{g})$. On the other hand, if

$$m \in \dot{\bigcup_{i=1}^{\lambda}} \mathrm{Cl}_N(m_i), \qquad m \in N - \Bigg(\dot{\bigcup_{i=1}^{\lambda}} \mathrm{Cl}_N(m_i) \Bigg)$$

then $|\mathrm{Cl}_{\bar G}(\bar g)\cap C_G(m)|\geqq 1$ and $|\overline{C_G(m^n)}|=|\overline{C_G(m)^n}|=|\overline{C_G(m)}^{\bar n}|=|\overline{C_G(m)}|$ yields

$$\sum_{n\in N}|\mathrm{Cl}_{\bar G}(\bar g)\cap\overline{C_G(n)}|\cdot|C_N(n)|=\sum_{i=1}^{\lambda}\sum_{n\in \mathrm{Cl}_N(m_i)}|\mathrm{Cl}_{\bar G}(\bar g)\cap\overline{C_G(m_i)}|\,|C_N(m_i)|$$

$$=\sum_{i=1}^{\lambda}|\mathrm{Cl}_{\bar G}(\bar g)\cap\overline{C_G(m_i)}|\cdot|\mathrm{Cl}_N(m_i)|\cdot|C_N(m_i)|=\sum_{i=1}^{\lambda}|\mathrm{Cl}_{\bar G}(\bar g)\cap\overline{C_G(m_i)}|\,|N|$$

and the desired formula is got directly from (1).

COROLLARY 6. *Let $N\trianglelefteq G$ and $g\in G$. Then*

(1) $r_G(gN)\leqq|\mathrm{Cl}_{\bar G}(\bar g)|\cdot r_G(N)$ *and we have the equality iff* $\bar g\in\bigcap_{x\in G}\mathrm{Core}\,\overline{C_G(x)}$.

(2) $r(G)\leqq|G/N|\cdot r_G(N)$ *and we have the equality iff* $C_G(x)N=G$ *for every* $x\in G$.

(3) $r_G(gN)\leqq|G'N/N|\cdot r_G(N)$, *where* $G'=[G,G]$ *denotes the derived subgroup of* G.

(4) *There exists* $z\in G$ *such that* $r(G)\leqq|\mathrm{Cl}_{\bar G}(\bar z)|\cdot r_G(N)\cdot r(G/N)$.

PROOF. (1) In the formula (1), we notice that $|\mathrm{Cl}_{\bar G}(\bar g)\cap\overline{C_G(x)}|\leqq|\mathrm{Cl}_{\bar G}(\bar g)|$, therefore

$$r_G(gN)\leqq(1/|G|)\cdot\sum_{x\in G}|\mathrm{Cl}_{\bar G}(\bar g)|\cdot|C_N(x)|=|\mathrm{Cl}_{\bar G}(\bar g)|\cdot r_G(N).$$

Moreover, we have the equality iff $\mathrm{Cl}_{\bar G}(\bar g)\subseteq\overline{C_G(x)}$ for every $x\in G$, iff $\bar g\in\overline{C_G(x)}^{\bar z}$ for every $x,z\in G$, that is, $\bar g\in\bigcap_{x\in G}\mathrm{core}\,\overline{C_G(x)}$.

(2) We have $r(G)=r_G(N)+\Sigma_{i=2}^{t}r_G(g_iN)$, with $\{\bar g_1=\bar 1,\bar g_2,\dots,\bar g_t\}$ a system of representatives from the conjugacy classes of $\bar G=G/N$. Therefore,

$$r(G)\leqq r_G(N)+|\mathrm{Cl}_{\bar G}(\bar g_2)|r_G(N)+\cdots+|\mathrm{Cl}_{\bar G}(\bar g_t)|r_G(N)=|G/N|\cdot r_G(N).$$

Moreover, the equality holds, if and only if $\bar g\in\bigcap_{x\in G}\mathrm{Core}\,\overline{C_G(x)}$ for every $g\in G-N$, that is, $\overline{C_G(x)}=\bar G$ for every $x\in G$.

(3) It follows from (1), because $|\mathrm{Cl}_{\bar G}(\bar g)|\leqq|\bar G'|=|G'N/N|$.

(4) It is enough to choose $z\in G$ such that

$$r_G(zN)=\max\{r_G(g_iN)\mid i=1,\dots,t\}.$$

Next, we obtain some information about the number $r_G(gN)$ in a different way; we analyze the congruence class of $r_G(gN)$ modulo the number

$$d=\mathrm{g.c.d.}(p_1-1,\dots,p_a-1)$$

where $|G| = p_1^{b_1} \cdots p_a^{b_a}$ is the factorization of order of G in primary powers. We have:

THEOREM 2. *Let $N \trianglelefteq G$ and $g \in G$. Then* $r_G(gN) \equiv 1 \pmod d$.

PROOF. Set $T_g = \dot{\bigcup}_{j=1}^{e} g^{y_j}N$, where $\{\bar{y}_1, \ldots, \bar{y}_e\}$ is a right transversal of $C_{\bar{G}}(\bar{g})$ in $\bar{G}$. We have

$$|T_g| = \sum_{j=1}^{e} |g^{y_j}N| = e \cdot |N| = |\mathrm{Cl}_{\bar{G}}(\bar{g})| \cdot |N|.$$

On the other hand, set $T_g = \mathrm{Cl}_G(xn_1) \dot{\cup} \cdots \dot{\cup} \mathrm{Cl}_G(xn_s)$, with $s = r_G(gN)$. Then $|N| \cdot |\mathrm{Cl}_{\bar{G}}(\bar{g})| = \Sigma_{i=1}^{s} |\mathrm{Cl}_G(xn_i)|$. Since the numbers that appear in the previous equality are congruent with 1 modulo d, we conclude that $1 \equiv s \pmod d$.

EXAMPLES. (1) If $|G|$ is an odd number, then $r_G(gN) \equiv 1 \pmod 2$ for each N normal subgroup of G and each element $g \in G$.

(2) If G is a p-group and $N \trianglelefteq G$, then $r_G(gN) \equiv 1 \pmod{p-1}$ for each $g \in G$.

2. The tuple $\Delta_{T_g}^G$

In this section we are going to study the tuples $\Delta_{T_g}^G$ in diverse special situations.

If S is a non-empty subset of G, $\mathcal{P}_j(S)$ denotes the set of all j-th powers of elements of S, that is, $\mathcal{P}_j(S) = \{s^j \mid s \in S\}$. Naturally, $\mathcal{P}_j(S)^x = \mathcal{P}_j(S^x)$ for every $x \in G$. We have:

THEOREM 3. *Let $N \trianglelefteq G$ and let g be an element of $G - N$. Let j be an integral number satisfying the following conditions*: g.c.d.$(j, |G|) = 1$ *and* $\mathcal{P}_j(gN) = g^jN$. *Then* $r_G(g^jN) = r_G(gN)$ *and* $\Delta_{T_{g^j}}^G = \Delta_{T_g}^G$.

PROOF. We have

$$T_{g^j} = \bigcup_{x \in G} (g^jN)^x = \bigcup_{x \in G} \mathcal{P}_j(gN)^x = \bigcup_{x \in G} \mathcal{P}_j((gN)^x) = \mathcal{P}_j\Big(\bigcup_{x \in G} g^xN\Big) = \mathcal{P}_j(T_g).$$

Therefore, if

$$T_g = \dot{\bigcup_{i=1}^{s}} \mathrm{Cl}_G(gn_i),$$

$\mathcal{P}_j(T_g) = T_{g^j}$ yields $T_{g^j} = \bigcup_{i=1}^{s} \mathrm{Cl}_G((gn_i)^j)$. Further, as j is coprime with $|G|$, it follows that the application ψ: $G \to G$, defined by $\psi(g) = g^j$, is a bijection. Consequently, $\mathrm{Cl}_G(z_1^j) = \mathrm{Cl}_G(z_2^j)$ iff $\mathrm{Cl}_G(z_1) = \mathrm{Cl}_G(z_2)$ and also $C_G(z^j) = C_G(z)$ for any $z_1, z_2, z \in G$. Thus $T_{g^j} = \dot{\bigcup}_{i=1}^{s} \mathrm{Cl}_G((gn_i)^j)$ and we get the desired result.

COROLLARY 7. *Let $N \trianglelefteq G$ and let g be an element of G. Then* $r_G(gN) = r_G(g^{-1}N)$ *and* $\Delta_{T_g}^G = \Delta_{T_{g^{-1}}}^G$.

PROOF. It is an immediate consequence from Theorem 3.

EXAMPLES. (1) Assume G is a group of odd order. Then we know that $r_G(g_iN) \equiv 1 \pmod 2$. Moreover, $\bar{g}_i$ is non-conjugate to $\bar{g}_i^{-1}$ in $\bar{G}$, therefore from (1), we get

$$r(G) \equiv r_G(N) + 2(t-1)/2 = r_G(N) + r(G/N) - 1 \pmod 4,$$

grouping into the sum every $r_G(zN)$ with $r_G(z^{-1}N)$ for each $z \in G - N$.

(2) Let G be a p-group, $N \leqq Z(G)$ and j an integral number such that $p \nmid j$, then $r_G(g^jN) = r_G(gN)$ and $\Delta^G_{T_{g^j}} = \Delta^G_{T_g}$.

Now let's consider the case $G/N \simeq C_p$, with p a prime number. Let $G/N = \langle \bar{g} \rangle \simeq C_p$. Burnside proves that $r_G(g^jN) = r_G(gN) = s$ for each $j = 1, 2, \ldots, p-1$ and $r(G) = ps + (r(N) - s)/p$. Nevertheless, he doesn't give any information about the conjugacy vector Δ_G. We have:

THEOREM 4. *Let $N \trianglelefteq G$ be such that $G/N = \langle \bar{g} \rangle \simeq C_p$, with p a prime number. We have*:

(1) *If p is the least prime divisor of $|G|$, then $\Delta^G_{gN} = \Delta^G_{g^jN} = (r_1, \ldots, r_s)$ for each $j = 1, \ldots, p-1$ and the $r(G)$-tuple*

$$(p\,|C_N(n_1)|, \ldots, p\,|C_N(n_s)|, |C_N(m_1)|, \overset{t=(r(N)-s)/p}{\ldots\ldots}, |C_N(m_t)|, r_1, \overset{p-1}{\ldots}, r_1, \ldots, r_s, \overset{p-1}{\ldots}, r_s)$$

where $\mathrm{Cl}_N(n_1), \ldots, \mathrm{Cl}_N(n_s)$ are the conjugacy classes of N fixed by g and $\mathrm{Cl}_G(m_1), \ldots, \mathrm{Cl}_G(m_t)$, the rest of the G-conjugacy classes of N, is the tuple conjugacy vector except the components order.

(2) *In general if $\Delta^G_{g^jN} = (r_{j1}, \ldots, r_{js})$ such that $r_{j1} \geqq \cdots \geqq r_{js}$ for every $j = 1, \ldots, p-1$, then $r_{11} = \cdots = r_{p-11}$, and $r_{1s} = \cdots = r_{p-1s}$.*

PROOF. (1) We know that $r_G(g^jN) = s$ for $j = 1, \ldots, p-1$. Set

$$gN = T_g = \mathrm{Cl}_G(gn_1) \,\dot{\cup} \cdots \dot{\cup}\, \mathrm{Cl}_G(gn_s).$$

We have $\mathrm{Cl}_G((gn_1)^j) \cup \cdots \cup \mathrm{Cl}_G((gn_s)^j) \subseteq T_{g^j} = g^jN$. Since $1 \leqq j \leqq p-1 < p$ and p is the least prime divisor of $|G|$, it follows that g.c.d.$(j, |G|) = 1$ therefore the previous union is disjoint, but $r_G(T_{g^j}) = s$, so necessarily

$$T_{g^j} = g^jN = \mathrm{Cl}_G((gn_1)^j) \,\dot{\cup} \cdots \dot{\cup}\, \mathrm{Cl}_G((gn_s)^j)$$

and now it's obvious that $\Delta^G_{g^jN} = \Delta^G_{gN}$.

(2) Let's fix j. Let $r_{j1} = |C_G(g_{j1})|$ and $r_{js} = |C_G(g_{js})|$. We have $\bar{G} = \langle \bar{g}_{11} \rangle = \langle \bar{g}_{j1} \rangle \simeq C_p$, so there exists $l \in \mathbf{N}$ such that $g_{j1}N = (g_{11}N)^l = g_{11}^lN$, therefore

$C_G(g_{11}) \leqq C_G(g_{11}^l)$ yields

$$r_{11} = |C_G(g_{11})| \leqq |C_G(g_{11}^l)| = r_{ji} \qquad \text{for some } i.$$

Similarly, there exists a natural number l' such that $g_{11}N = (g_{j1}N)^{l'} = g_{j1}^{l'}N$ and $C_G(g_{j1}) \leqq C_G(g_{j1}^{l'})$ implies $r_{j1} = |C_G(g_{j1})| \leqq |C_G(g_{j1}^{l'})| = r_{1i'}$ for some i'. Thus, $r_{11} \leqq r_{ji} \leqq r_{j1} \leqq r_{1i'}$, but $r_{11} \geqq r_{1i'}$, so all of them are equal and therefore $r_{11} = r_{j1}$.

A similar reasoning proves that $r_{1s} = \cdots = r_{p-1s}$.

EXAMPLES. (1) If $s \leqq 3$, the equation $1/p = \sum_{i=1}^{s} 1/r_{ji}$, $j = 1, \ldots, s$ forces $\Delta_{g^jN}^G = \Delta_{gN}^G$ for $j = 1, \ldots, p-1$.

(2) If G is a p-group of order p^a and $G/N \simeq C_p$, then

$$\Delta_{g^jN}^G = (p^{t_1}, \ldots, p^{t_s}) \qquad \text{for each } j = 1, 2, \ldots, p-1,$$

satisfying the following conditions: $t_1 \geqq \cdots \geqq t_s$ and $1/p = \sum_{i=1}^{s} 1/p^{t_i}$.

In [4] we determined Δ_G completely, in the case $G = Nx_\lambda K$, with N an abelian group and supposing that the action λ of K in $\mathrm{Aut}(N)$ is known. In general, if N is non-abelian, we observe the following:

PROPOSITION 1. *Let $G = N \times_\lambda K$. If $g \in \mathrm{Ker}\,\lambda \cap Z(K)$, then $r_G(gN) = r_G(N)$ and $\Delta_{T_g}^G = \Delta_N^G$.*

PROOF. We have $g \in K \cap Z(G)$. Now the result is obvious because, as G is a semidirect product, we have $T_g = gN = g\,\mathrm{Cl}_G(n_1) \dot{\cup} \cdots \dot{\cup}\, g\,\mathrm{Cl}_G(n_s)$ if $N = \bigcup_{i=1}^{s} \mathrm{Cl}_G(n_i)$.

We have also:

PROPOSITION 2. *Let $\alpha \in \mathrm{Aut}(G)$ and $N \trianglelefteq G$ such that $N^\alpha = (N)\alpha = N$. Then $r_G(g^\alpha N) = r_G(gN)$ and $\Delta_{T_{g^\alpha}}^G = \Delta_{T_g}^G$.*

PROOF. We have $T_g^\alpha = \bigcup_{x \in G} (g^\alpha N)^{x^\alpha} = \bigcup_{y \in G} (g^\alpha N)^y = T_{g^\alpha}$, $\mathrm{Cl}_G(z_1)^\alpha = \mathrm{Cl}_G(z_2)^\alpha$ iff $\mathrm{Cl}_G(z_1) = \mathrm{Cl}_G(z_2)$, and $|\mathrm{Cl}_G(z^\alpha)| = |\mathrm{Cl}_G(z)|$. Therefore $r_G(g^\alpha N) = r_G(gN)$ and $\Delta_{T_{g^\alpha}}^G = \Delta_{T_g}^G$.

REFERENCES

1. W. Burnside, *Theory of Groups of Finite Order*, 2nd edn., Dover, 1955.
2. A. Vera López, *Conjugacy classes in finite solvable groups*, Isr. J. Math. **47** (1984), 241–245.
3. A. Vera López, *The number of conjugacy classes in a finite nilpotent group*, Rend. Sem. Mat. Univ. Padova **73** (1985), 209–215.
4. A. Vera López and J. Vera López, *Classification of finite groups according to the number of conjugacy classes*, Isr. J. Math. **51** (1985), 305–338.
5. A. Vera López and J. Vera López, *Classification of finite groups according to the number of conjugacy classes II*, Isr. J. Math. **56** (1986), 188–221 (this issue).